MIDTERM: ALGEBRAIC NUMBER THEORY

Date: **17th Feburary 2017**
The total points is **115** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (5+15=20 points) State Nakayama's lemma. Let (R, m) be a local ring and M be a finitely generated R-module. Show that M can be generated by $\dim_{R/m}(M/mM)$ elements as an R-module.
(2) (5+15=20 points) Let A be an integral domain. When is A said to be a normal domain? Let $A \subset B$ be domains and $\alpha \in B$ be integral over A. Let K be the field of fractions of A. Show that the minimal polynomial of α over K have coefficients in A.
(3) (5+15=20 points) Define Dedekind domain. Let R be Dedikind domain and S be a multiplicative subset of R. Show that $S^{-1}R$ is a field or a Dedikind domain.
(4) (5+15=20 points) Define class group of a Dedikind domain. Let R be a Dedikind domain with finitely many maximal ideals. Compute the class group of R.
(5) (10+10+15=35 points) Let $L = \mathbb{Q}(\sqrt{10})$ and R be the integral closure of $\mathbb{Z}$ in L.

    (a) Find the number of prime ideals of R containing 3 and compute the relative degree of these primes over $\mathbb{Z}$.
    (b) Find the number of prime ideals of R containing 7 and compute the relative degree of these primes over $\mathbb{Z}$.
    (c) Let p be a prime number different from 2 and 5. Show that the ramification index of any prime ideal P of R containing p is 1 (i.e. $e(P/p) = 1$).